Apparent superluminal propagation of a laser pulse in a dispersive medium

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The distortion of a laser pulse propagating in a dispersive gain/absorptive medium is analyzed. The relationship between the distortion of the pulse and superluminal propagation is discussed. We present an analytical approach based on the laser envelope equation that is readily applicable to arbitrary input pulse shapes. This analysis is used to interpret recent experiments that claim to have observed distortionless superluminal laser pulse propagation.

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I. INTRODUCTION

It is well known that in regions of anomalous dispersion the group velocity of an electromagnetic pulse can be abnormal, i.e., greater than c (the speed of light in vacuum) or negative [1,2]. While it has been claimed that group velocity "is just not a useful concept" in regions of strong anomalous dispersion [2], others have shown that for a Gaussian pulse the group velocity represents the velocity of the peak of the pulse even when it is abnormal [3–5]. This apparent superluminal propagation results from a pulse reshaping effect by which a dispersive medium preferentially amplifies the front or absorbs the back of the pulse. This effect has been described theoretically using a Fourier transform method. For analytical tractability a Gaussian pulse was used and the refractive index expanded to keep only the lowest-order group velocity dispersion (GVD) term [3–5].

The analysis presented here is based on an envelope equation that describes the propagation of arbitrary pulse shapes and can in principle include dispersive effects analytically to all orders [6]. For a pulse with a well-defined leading edge, we show that the lowest-order effect in a gain medium is that the pulse propagates with velocity c and undergoes a distortion in which the front of the pulse is amplified more than the back, i.e., *differential gain*. This leads to apparent superluminal pulse propagation in which the peak of the pulse travels faster than c. However, the velocity of the leading edge of the pulse does not exceed c. A related effect can also take place in an absorptive medium.

II. ANALYTICAL MODEL

The following analysis considers a laser pulse propagating through a general dispersive medium as illustrated in Fig. 1. The laser electric field E is described by the wave equation,

$$(\partial^2/\partial z^2 - c^{-2}\partial^2/\partial t^2)E = 4\pi c^{-2}\partial^2 P/\partial t^2.$$

The polarization field P is related to the electric field by

$$P(z,t) = (2\pi)^{-1/2}\int_{\infty}^{0} d\tau\, \chi(\tau) E(z,t-\tau),$$

where χ is the susceptibility. Defining the Fourier transform of a quantity $Q(z,t)$ as

$$\hat{Q}(z,\omega) = (2\pi)^{-1/2}\int_{-\infty}^{\infty} Q(z,t)\exp(i\omega t)\,dt,$$

the Fourier transforms of the polarization and the electric field are related by $\hat{P}(z,\omega) = \hat{\chi}(\omega)\hat{E}(z,\omega)$. The medium is characterized by a frequency-dependent complex refractive index $n(\omega) = \sqrt{1+4\pi\hat{\chi}(\omega)}$. We assume that the deviation of the refractive index from unity, i.e., $\Delta n(\omega) = n(\omega) - 1$, is small and neglect reflections of the laser pulse from the medium boundaries at $z=0$ and $z=L$. To determine the evolution of the pulse envelope we represent the laser electric field as $E(z,t) = \frac{1}{2}A(z,t)\exp[i(k_0 z - \omega_0 t)] + \text{c.c.}$, where $A(z,t)$ is the slowly varying, complex pulse envelope, $k_0(\omega_0) = \omega_0 n(\omega_0)/c$ is the complex wave number, ω_0 is the carrier frequency, and c.c. denotes the complex conjugate. The field is polarized in the transverse direction and propagates in the z direction. Since k_0 is complex, the factor $\exp[-\mathrm{Im}(k_0)z]$ represents an overall amplification/absorption of the pulse at frequency ω_0 and does not cause pulse distortion. The actual laser pulse amplitude is $|A(z,t)|\exp[-\mathrm{Im}(k_0)z]$.

An envelope equation is obtained by substituting the representation for the laser electric field into the wave equation and performing a spectral analysis [6–8] that involves expanding the refractive index about the carrier frequency ω_0.

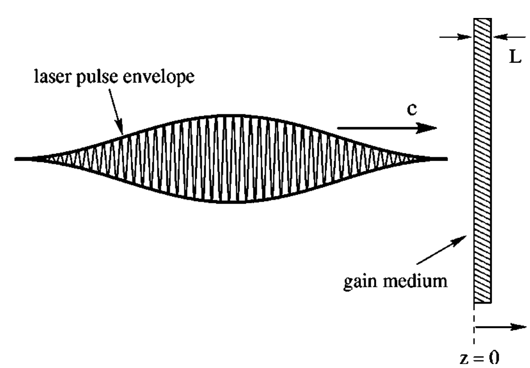

FIG. 1. Schematic showing a long laser pulse entering a gain medium.

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The envelope equation describing the evolution of the laser pulse, including dispersive effects to all orders, is given by

$$\left(\frac{\partial}{\partial z}+\frac{1}{c}\frac{\partial}{\partial t}\right)A(z,t) = -\left[\frac{\partial(\beta-\omega/c)}{\partial\omega}\right]_0\frac{\partial}{\partial t}A(z,t) + \frac{i}{2k_0}\left\{\frac{\partial^2}{\partial z^2}+\sum_{m=1}^{\infty}\frac{i^m}{m!}\left[\frac{\partial^m\beta^2}{\partial\omega^m}\right]_0\frac{\partial^m}{\partial t^m}\right\}A(z,t), \tag{1}$$

where $\beta(\omega)=\omega n(\omega)/c$ is the frequency-dependent wave number, $[\ ]_0$ denotes that the quantity in brackets is to be evaluated at $\omega=\omega_0$, and the laser pulse envelope at the input to the amplifying medium, $A(z=0,t)$, is assumed given. If the spectral width of the pulse is sufficiently narrow, it is valid to limit the analysis to terms of order $\partial^2/\partial t^2$, i.e., lowest-order GVD effects. With this approximation, together with neglecting the small term proportional to $\partial^2/\partial z^2$, Eq. (1) reduces to

$$\left(\frac{\partial}{\partial z}+\frac{1}{c}\frac{\partial}{\partial t}\right)A(z,t)=-\left(\kappa_1\frac{\partial}{\partial t}+\frac{i}{2}\kappa_2\frac{\partial^2}{\partial t^2}+\cdots\right)A(z,t), \tag{2}$$

where $\kappa_l=[\partial^l\kappa(\omega)/\partial\omega^l]_0$, $l=1,2,\ldots$, and $\kappa(\omega)=\omega\Delta n(\omega)/c$. This approximation, which requires both a sufficiently short interaction length and long pulse duration, is sufficient for the present purpose. For pulse propagation in vacuum, $\Delta n(\omega)=0$ so that the right-hand side (RHS) of Eq. (2) vanishes and the laser envelope is given by $A(z,t)=A(0,t-z/c)$, indicating that the pulse propagates with velocity c without distortion.

Equation (2) can be solved iteratively assuming that terms on the RHS get progressively smaller. However, to indicate where inconsistencies in the ordering of approximations can arise we proceed with a spectral analysis and show how to recover a consistent ordering. Equation (2) is Fourier transformed in time and the resulting differential equation in z is solved for the transformed envelope. Inverting the transformed envelope yields the solution

$$A(z,t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}d\nu\,\hat{A}(0,\nu)\times\exp[-i\nu(t-z/c)]\exp(i\kappa_1\nu z+i\kappa_2\nu^2 z/2), \tag{3}$$

where $\hat{A}(0,\nu)$ is the Fourier transform of the envelope at $z=0$, and ν is the transform variable.

It is assumed that the following inequalities hold: $1\gg|\kappa_1\nu z|\gg|\kappa_2\nu^2 z/2|$, where $\nu\approx 1/T$, $z\approx L$, and T is the characteristic pulse duration. To correctly evaluate the integral in Eq. (3), the exponentials in the small quantities should be expanded to an order of approximation consistent with Eq. (2), otherwise unphysical solutions may result. For example, if the lowest-order GVD term $\kappa_2\nu^2 z/2$ is neglected in Eq. (3), the laser envelope is given by

$$A(z,t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}d\nu\,\hat{A}(0,\nu)\exp[-i\nu(t-z/c)]\exp(i\kappa_1\nu z). \tag{4}$$

Equation (4) can be integrated exactly to give

$$A(z,t)=A(0,t-z/v_g), \tag{5}$$

where the quantity $v_g=[\partial(\omega n/c)/\partial\omega]^{-1}=c/(1+c\kappa_1)$ defines the group velocity of the pulse. The exact solution, given by Eq. (5), to the approximate envelope equation can clearly lead to unphysical results since it implies that, to the lowest order of approximation, the entire pulse propagates undistorted with velocity v_g. This interpretation is due to implicitly retaining terms beyond the order of the approximation through the exponential factor. For example, neglecting κ_2 terms in the exponent of Eq. (4) is equivalent to keeping terms proportional to $(\kappa_1\nu z)^2$ while neglecting terms proportional to $\kappa_2\nu^2 z$ that are of the same order.

In a dispersive gain/absorptive medium, v_g can be abnormal. For example, if $-1<c\kappa_1<0$, the pulse velocity exceeds c. A negative group velocity implies superluminal propagation if one considers the pulse delay time [9,10]. The delay time, $\Delta T=L/v_g-L/c$, is defined as the difference in the transit times of an arbitrary point on the pulse in the dispersive gain/absorptive medium and in vacuum. A negative delay time implies superluminal propagation. While it is physically possible for some points on the pulse to have negative delay times, e.g., the peak of the pulse, this should not be interpreted as superluminal propagation of the *entire* pulse since the pulse distorts.

To properly describe higher-order effects it is necessary to solve Eq. (3) by keeping the order of approximation consistent. Expanding the exponential terms in Eq. (3) to second order yields

$$A(z,t)=(1/\sqrt{2\pi})\int_{-\infty}^{\infty}d\nu\,\hat{A}(0,\nu)[1+i\kappa_1 z\nu+(1/2)(i\kappa_2 z-\kappa_1^2 z^2)\nu^2]\times\exp[-i\nu(t-z/c)]. \tag{6}$$

Equation (6) can be integrated to give

$$A(z,t)=\left(1-\kappa_1 z\frac{\partial}{\partial t}-\tfrac{1}{2}(i\kappa_2 z-\kappa_1^2 z^2)\frac{\partial^2}{\partial t^2}+\cdots\right)\times A(0,t-z/c). \tag{7}$$

In Eq. (7) the first term on the RHS denotes the vacuum solution, the second term represents lowest-order differential gain, while the third- and higher-order terms are small and denote higher-order effects. Equation (7) shows that the pulse propagates at the speed of light while undergoing differential gain (distortion). The quantity κ_1 can be negative in the presence of gain or absorption. In the case of gain, when

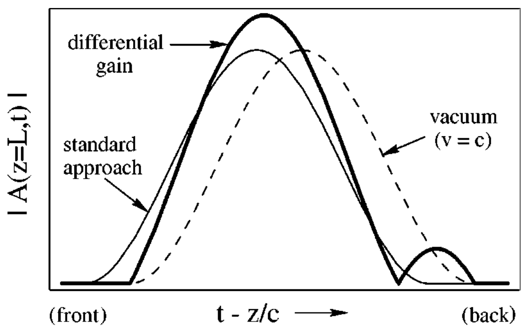

FIG. 2. Illustrations of the pulse shape at the exit boundary, comparing vacuum propagation (dashed curve), pulse advancement obtained by Eq. (5) (light solid curve), and pulse distortion due to differential gain given by Eq. (7) (thick curve).

$\kappa_1 < 0$, the front portion of the pulse is amplified more than the back. Figure 2 is an illustration comparing the type of pulse distortion characteristic of differential gain ($\kappa_1 < 0$) with the distortionless pulse advancement described by Eq. (5). As shown in the following section, pulse distortion similar to that exhibited by the darker curve of Fig. 2 results from the exact numerical solution of the wave equation.

It is interesting to note that for a Gaussian pulse, the integral in Eq. (3) can be evaluated exactly if the expansion of the refractive index is carried up to κ_2, i.e., lowest-order GVD. Taking the input laser pulse to have the form $A(0,t) = a_0 \exp(-t^2/2T^2)$, where a_0 is the peak amplitude, the Fourier transform is $\hat{A}(0,\nu) = a_0 T \exp(-\nu^2 T^2/2)$. For this pulse form, the integral in Eq. (3) can be evaluated to give [4]

$$A(z,t) = \frac{a_0}{\sqrt{1 - i\kappa_2 z/T^2}} \exp\left\{ \frac{-[t-(1+c\kappa_1)z/c]^2}{2T^2(1-i\kappa_2 z/T^2)} \right\}, \quad (8)$$

where $\mathrm{Re}(1 - i\kappa_2 z/T^2) > 0$, i.e., $-\mathrm{Im}(\kappa_2) z/T^2 < 1$ is required for convergence of the integral. This analysis shows that the pulse propagates with velocity v_g (even if v_g is negative or greater than c) and remains Gaussian but with a different amplitude and width. This result is specific to a Gaussian pulse, which does not have a well-defined beginning or end [3,4]. Using Eq. (7), however, a Gaussian pulse does not remain Gaussian, but becomes distorted according to

$$A(z,t) = a_0 \left[1 + \kappa_1 z \frac{\tau}{T^2} - \frac{1}{2T^4} (i\kappa_2 z - \kappa_1^2 z^2)(\tau^2 - T^2) + \cdots \right] \times \exp(-\tau^2/2T^2),$$

where $\tau = t - z/c$.

The energy gain/loss can also be calculated to all orders from the pulse envelope. The ratio of the input pulse energy to the pulse energy within the medium is found to be given by

$$\frac{\varepsilon(z)}{\varepsilon(0)} = \exp[-2\,\mathrm{Im}(k_0) z] \int_{-\infty}^{\infty} |A(z,t')|^2\, dt' \times \left(\int_{-\infty}^{\infty} |A(0,t')|^2 dt' \right)^{-1}. \quad (9)$$

Using Eq. (7), Eq. (9) can be integrated for an arbitrary pulse shape from $t' = 0$ to $t' = T$ to give

$$\frac{\varepsilon(z)}{\varepsilon(0)} = \exp[-2\,\mathrm{Im}(k_0) z][1 - \{\mathrm{Im}(\kappa_2) z - [2\,\mathrm{Im}(\kappa_1) z]^2\} \pi^2/3T^2]. \quad (10)$$

The exponential factor in Eq. (10) represents the homogeneous gain of the medium while the higher-order terms in the bracket represent dispersion effects.

III. NUMERICAL RESULTS AND COMPARISON WITH EXPERIMENTS

In a recent article [10], titled *Gain-assisted superluminal light propagation*, researchers reported observing superluminal propagation of a laser pulse through an amplifying medium by a new mechanism that does not distort the pulse. In this experiment, a long laser pulse was passed through an amplifying medium consisting of a specially prepared caesium gas cell of length $L = 6$ cm, as depicted in Fig. 1. The laser pulse of duration $T = 3.7\,\mu$ sec was much longer (1.1 km) than the gas cell, so that at any given instant only a small portion of the pulse was inside the cell. By measuring the pulse amplitude at the exit, it is claimed that both the front and the back edges of the pulse were shifted forward in time by the same amount relative to a pulse that propagated through vacuum. In contrast to earlier works that have interpreted apparent superluminal propagation as a pulse reshaping effect [3,4], it is claimed in Ref. [10] that superluminal propagation is observed ''while the shape of the pulse is preserved'' and ''the argument that the probe pulse is advanced by amplification of its front edge does not apply.'' This article generated a great deal of press attention around the world.

The results of our analysis is used to interpret the experiment of Ref. [10]. To proceed, we consider the following standard model for the refractive index of a multiline gain medium [11],

$$n^2(\omega) = 1 + \sum_i \frac{(4\pi N_i q^2/m)(\rho_{u,i} - \rho_{g,i})}{\omega^2 - \Omega_i^2 + 2i\omega\gamma_i}, \quad (11)$$

where the sum is over atomic levels, and $\rho_{u,i}$ and $\rho_{g,i}$ are the density matrix elements for the excited and ground states, respectively. The quantities Ω_i and γ_i denote the resonant frequency and line width of the ith level. An inverted population or gain medium is characterized by $(4\pi N q^2/m)(\rho_u - \rho_g)/4\Omega^2 > 0$. To model the experiment of Ref. [10], the following near-resonance two-level approximation of Eq. (11) is employed:

$$\hat{\chi}(f) \cong \frac{\Delta n(\omega)}{2\pi} = \frac{M_1}{f - f_1 + i\gamma} + \frac{M_2}{f - f_2 + i\gamma}, \quad (12)$$

where $M_{1,2} > 0$ are related to the gain coefficients. The susceptibility in Eq. (12) represents a medium with two gain lines of spectral width γ at resonance frequencies f_1 and f_2. The gain spectrum for $M_{1,2} = M = 0.18$ Hz, f_1

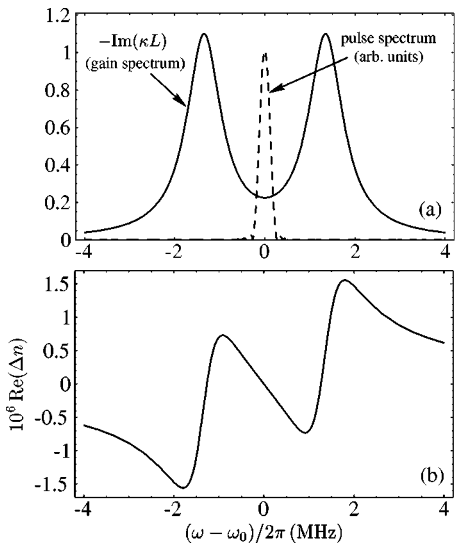

FIG. 3. Gain spectrum (solid curve) obtained using the susceptibility in Eq. (9) for the parameters $M=0.18$ Hz, $f_1=3.5\times10^{14}$ Hz, $f_2=f_1+2.7$ MHz, and $\gamma=0.46$ MHz. The dashed curve shows the spectrum associated with the pulse envelope of Eq. (10) with $T=3.7$ μsec.

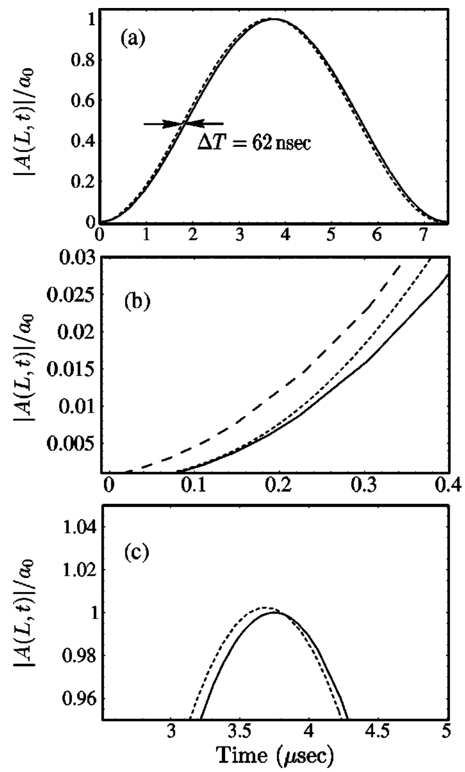

FIG. 4. Dotted curves show the pulse envelope amplitude $|A(L,t)|$ at $z=L$ obtained from Eq. (7). Solid curves denote a pulse that has traveled a distance L through vacuum. The dashed curve in panel (b) is the unphysical solution obtained from Eq. (5) showing superluminal propagation. Panels (b) and (c) are expanded views of the front and peak of the pulse, respectively. The parameters for this figure are the same as in Fig. 2.

$=3.5\times10^{14}$ Hz, $f_2=f_1+2.7$ MHz and $\gamma=0.46$ MHz, is shown in Fig. 3(a) (solid curve). For these parameters the deviation of the refractive index from unity $\Delta n(\omega)$ shown in Fig. 3(b) closely approximates that in Fig. 3 of Ref. [10]. The input laser pulse envelope is taken to have the form

$$A(z=0,t)=\begin{cases} a_0\sin^2(\pi t/2T), & 0<t<2T \\ 0, & \text{otherwise,}\end{cases} \tag{13}$$

where a_0 is the pulse amplitude and $\omega_0/2\pi=(f_1+f_2)/2$ is the carrier frequency. The spectrum associated with the input pulse is shown by the dashed curve in Fig. 3 and has no significant spectral components at the gain lines.

For the parameters of Ref. [10] we find that the first-order correction in Eq. (7), i.e., the term proportional to $\partial/\partial t$, is of order $\kappa_1 L/T\approx-1.6\times10^{-2}$ giving a negative group velocity $v_g=-c/310$. The second order correction is $\kappa_2 L/T^2\approx-10^{-3}i$. Hence, the expansion performed to obtain Eq. (7) is valid. The differential gain effect, in which the peak of the pulse is advanced, requires that $\kappa_1<0$. Using Eq. (13) we find that κ_1 is approximately given by

$$c\kappa_1\cong-8\pi\frac{(f_1+f_2)}{(f_2-f_1)^2}M, \tag{14}$$

where it has been assumed that $|f-f_{1,2}|\gg\gamma$. In this case it is clear that a gain medium ($M>0$) is required for κ_1 to be negative. For this case, the gain coefficient is given by $-\mathrm{Im}(k_0)=8M\gamma/(f_2-f_1)^2>0$. Note that in an absorptive medium ($M<0$), κ_1 can also be negative provided $|f-f_{1,2}|\ll\gamma$. In this case differential absorption occurs in which the back of the pulse is absorbed more than the front [3,4].

The validity of Eq. (7) was verified by numerically solving the envelope equation to all orders in κ_l. Figure 4 compares the solution given by Eq. (7) at the exit of the gain medium (dotted curves) with the vacuum solution $|A(0,t-L/c)|$ (solid curves). Panel (a) shows the entire pulse profile. Consistent with the experimental measurements, the leading edge is shifted forward in time relative to the vacuum solution by 62 nsec. Panel (b) shows three curves: the solid curve denotes the vacuum solution, the dotted curve shows the result obtained from Eq. (7), and the dashed curve shows the result obtained from Eq. (5). The dotted curve shows that the front of the pulse propagates with velocity c; the propagation is not superluminal. The unphysical solution, given by the dashed curve, shows the front of the pulse propagating at the superluminal group velocity. Panel (c) is an expanded view near the peak of the pulse showing that the front is amplified more than the back. This analysis indicates that differential gain occurred in the experiment of Ref. [10] and can account for the observed pulse advancement. Hence, the interpretation in Ref. [10] that superluminal propagation occurs without amplification of the leading edge of the pulse is incorrect.

IV. CONCLUSIONS

We have analyzed the propagation of a laser pulse in a dispersive gain/absorptive medium using an approach based on the pulse envelope equation. Using this approach allows analysis of higher-order dispersive effects and arbitrary input pulse shapes. We find that to properly describe pulse propagation, a consistent ordering of the approximations is necessary. We show that in a gain medium, the lowest-order effect is that the pulse propagates with velocity c and undergoes differential gain, i.e., a distortion in which the front of the pulse is amplified more than the back. Our analysis indicates that differential gain was responsible for the pulse advancement observed in the experiment of Ref. [10] and not a newly observed mechanism for superluminal propagation.

ACKNOWLEDGMENTS

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